# Hypothesis Testing: Z-Test

## Concepts

1. In general, statistics does not allow you to prove anything is true, but instead allows you to show that things are probably false. So when we do hypothesis testing, the **null hypothesis** $H_0$ is something that we want to show is false and the **alternative hypothesis** $H_1$ is something that you want to show is true. For example, to show that a drug cures cancer, the null hypothesis would be that the drug does nothing and the alternative hypothesis would be that the drug does help cure cancer.

   A **type 1 error** is rejecting a true null which means that in our example, saying a drug cures cancer when it doesn't. A **type 2 error** is failing to reject a false null which means in our case as saying that the drug doesn't do anything when it does. The **significance level** $\alpha$ is the probability of making a type 1 error. The **power** $1 - \beta$ is 1 minus the probability of making a type 2 error.

## Examples

2. When Thanos snapped his fingers, everyone had a $p = 0.5$ chance of disintegrating. I think that this probability was much lower for the original Avengers. Out of the 6 of them, no one got disintegrated. Can you reject the null hypothesis that there was a $p = 0.5$ chance of each of them disintegrating with an $\alpha = 0.05$?

   > **Solution:** Our null hypothesis is $H_0 : p = 0.5$ and alternate hypothesis $H_1 : p < 0.5$. We want the probability of at least that unlikely of a scenario which is $P(X \leq 0)$ less than or equal to 0 of them got disintegrated. This is a binomial distribution so the probability of this happening is $P(X \leq 0) = P(X = 0) = \frac{1}{2^6} = 0.015 < \alpha$. Therefore we can reject the null hypothesis and say that they were given some special treatment.

3. An infomercial claims that a miracle drug will cause you to grow all your hair back. There are 25 brave participants and surprisingly 7 people regrew their hair. If normally 10% of people regrow their hair, can you say that this drug worked?

**Solution:** We would expect that 10% of people will regrow their hair with standard deviation $\sigma = \sqrt{p(1-p)} = \sqrt{0.1(0.9)} = 0.3$. The central limit theorem says that with a sample of 25 people, we expect that 10% of people regrow their hair with a standard deviation of $\sigma/\sqrt{n} = 0.3/\sqrt{25} = 0.06$. There are $7/25 = 28\%$ who regrew their hair. The $z$ score is $z(|0.28 - 0.1|/0.06) = z(3) < \alpha$. Therefore, we can reject the null hypothesis and say that this drug does help you grow your hair back.

## Problems

4. (True story) A woman claimed that she could tell whether milk or tea was added first to a cup. She was given 4 cups with milk added first then tea and 4 cups of the opposite. She guessed all 8 correctly. Let the null hypothesis be that she guesses randomly and alternate hypothesis be that she actually can tell with $p > 0.5$. Can we say that she has this ability with $\alpha = 0.05$?

   **Solution:** The null hypothesis would be that she guessed randomly and the probability that she was successful once is $p = \frac{1}{2}$. We want to calculate the probability that she did at least this well so $P(X \geq 8) = P(X = 8) = \binom{8}{8}\frac{1}{2^8} = \frac{1}{2^8} = 0.004 < \alpha$. Therefore, we can reject the null hypothesis and say that she does have this ability.

5. You flip a coin 100 times and get 55 heads. Can you say that it is biased towards heads? (use $\alpha = 0.05$)

   **Solution:** The null hypothesis is that the coin is unbiased and hence $p = 0.5$. The standard deviation is $\sigma = \sqrt{p(1-p)} = 0.5$. Thus, the central limit theorem tells us that the percentage of coin flips we get is approximately normally distributed with standard deviation $0.5/\sqrt{n} = 0.5/\sqrt{100} = 0.05$. We got $55/100 = 0.55$ percent of heads. Calculating the $z$ score of 0.55 is $z(|0.55 - 0.5|/\sigma) = z(1) > \alpha$ and hence we cannot reject the null hypothesis.

6. An infomercial claims that a miracle drug will cause you to grow all your hair back. There are 100 brave participants and this time 20 people regrow their hair. If normally 10% of people regrow their hair, can you say that this drug worked?

   **Solution:** We would expect that 10% of people will regrow their hair with standard deviation $\sigma = \sqrt{p(1-p)} = \sqrt{0.1(0.9)} = 0.3$. The central limit theorem says that with a sample of 100 people, we expect that 10% of people regrow their hair with a standard deviation of $\sigma/\sqrt{n} = 0.3/\sqrt{100} = 0.03$. There are $20/100 = 20\%$ who

regrew their hair. The $z$ score is $z(|0.2 - 0.1|/0.03) = z(3.33) < \alpha$. Therefore, we can reject the null hypothesis and say that this drug does help you grow your hair back.